Estimates for eigenvalues of Schrödinger operators with complex-valued potentials

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Abstract

New estimates for eigenvalues of non-self-adjoint multi-dimensional Schrödinger operators are obtained in terms of L_p-norms of the potentials. The results extend and improve those obtained previously. In particular, diverse versions of an assertion conjectured by Laptev and Safronov are discussed. Schrödinger operators with slowly decaying potentials are also considered.

Keywords. Schrödinger operators; polyharmonic operators; complex potential; estimation of eigenvalues.

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1 Introduction

In this paper we discuss estimates for eigenvalues of Schrödinger operators with complex-valued potentials. Among existing results on this problem regarding non-self-adjoint Schrödinger operators we mention the works [AAD01], [FLS11], [Fra11], [Saf10a], [Saf10b], and also [Dav02] for an overview on certain aspects of spectral analysis of non-self-adjoint operators mainly needed for problems in quantum mechanics. In [AAD01] it was observed that for the one-dimensional Schrödinger operator $H = -d^2/dx^2 + q$, where the potential q is a complex-valued function belonging to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, every its eigenvalue λ which does not lie on the non-negative semi-axis satisfies the following estimate

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{-\infty}^{\infty} |q(x)|\, dx. \tag{1.1}$$

For the self-adjoint case the estimate (1.1) was pointed out previously by Keller in [Kel61]. In [FLS11] related estimates are found for eigenvalues of Schrödinger operators on semi-axis with complex-valued potentials. Note that, as is pointed out in [FLS11], the obtained estimates are in sense sharp for both cases of Dirichlet and Neumann boundary conditions. In [Fra11], [Saf10a] (see also [Saf10b]) the problem is considered for higher dimensions case. In particular, in [Fra11]

estimates for eigenvalues of Schrödinger operators with complex-valued potentials decaying at infinity, in a certain sense, are obtained in terms of appropriate weighted Lebesgue spaces norms of potentials.

In this paper we mainly deal with the evaluation of eigenvalues of multi-dimensional Schrödinger operators. The methods which we apply allow us to consider the Schrödinger operators acting in one of the Lebesgue space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$. We consider the formal differential operator $-\Delta + q$ on $\mathbb{R}^n$, where Δ is the n-dimensional Laplacian and q is a complex-valued measurable function. Under some reasonable conditions, ensuring, in a suitable averaged sense, decaying at infinity of the potential, there exists a closed extension H of $-\Delta+q$ in the space $L_p(\mathbb{R}^n)$ such that its essential spectrum $\sigma_{ess}(H)$ coincides with the semi-axis $[0, \infty)$, and any other point of the spectrum, i.e. not belonging to $\sigma_{ess}(H)$, is an isolated eigenvalue of finite (algebraic) multiplicity. We take the operator H as the Schrödinger operator corresponding to $-\Delta + q$ in above sense and we will be interested to find estimates of eigenvalues of H which lie outside of the essential spectrum. The problem reduces to estimation of the resolvent of the unperturbed operator H_0, that is defined by $-\Delta$ in $L_p(\mathbb{R}^n)$ on its domain the Sobolev space $W_p^2(\mathbb{R}^n)$, bordered by some suitable operators of multiplication (cf. reasoning in Section 2).

We begin with three dimensional Schrödinger operators (cf. Section 3). In this case the resolvent $R(\lambda; H_0)$ of H_0 is an integral operator with the kernel $exp(-\mu|x-y|)/4\pi|x-y|$, where $\mu = -i\lambda^{1/2}$, with for instance $\operatorname{Im}\lambda^{1/2} > 0$. Due to this fact the evaluation of the bordered resolvent of the unperturbed operator H_0 are made by applying direct standard methods. For the higher dimensional case the approach used in the proofs concerning Schrödinger operators on $\mathbb{R}^3$ is not so convenient to apply. Instead we propose other methods of obtaining bounds for eigenvalues. These methods involve somewhat *heat kernels* associated to the Laplacian (cf. Section 4). For it could be used the kernel $(4\pi it)^{-n/2}exp(-|x-y|^2/4it), -\infty < t < \infty$, representing the operator-group $U(t) = exp(-itH_0), -\infty < t < \infty$, and then making use of the formula expressing the resolvent $R(\lambda; H_0)$ as the Laplace transform of $U(t)$ (see [HP74]). In this way we obtain a series of estimates for perturbed eigenvalues. In particular, supposing that $q = ab$, where $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ for r, s satisfying $0 < r \le \infty$, $p \le s \le \infty$, $r^{-1} - s^{-1} = 1 - 2p^{-1}$, $2^{-1} - p^{-1} \le r^{-1} \le 1 - p^{-1}$ and $r^{-1} + s^{-1} < 2n^{-1}$, for any complex eigenvalue λ of the Schrödinger operator H with $\operatorname{Im}\lambda \neq 0$, we have

$$|\operatorname{Im}\lambda|^{\alpha} \le (4\pi)^{\alpha-1}\Gamma(\alpha)\|a\|_r\|b\|_s, \tag{1.2}$$

in which $\alpha := 1 - n(r^{-1} + s^{-1})/2$ (Γ denotes the gamma function). An immediately consequence of this result (letting $r = s = 2\gamma + n,\ \gamma > 0$) is the estimate

$$|\operatorname{Im}\lambda|^{\gamma} \le (4\pi)^{-n/2}\Gamma\left(\frac{\gamma}{\gamma + n/2}\right)^{\gamma+n/2} \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2}\, dx \tag{1.3}$$

for $n \ge 3, \gamma > 0$. The estimate (1.3) is a version of a conjecture due to Laptev and Safronov [LS09].

Estimation of eigenvalues can be made representing *a priori* the resolvent of H_0 in terms of Fourier transform (see Section 4 and 5). The method leads,

in particular, to the following result. Let $1 < p < \infty$, and let $q = ab$ with $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ for $0 < r, s \leq \infty$ satisfying $2^{-1} - p^{-1} \leq r^{-1} \leq 1 - p^{-1}$, $-2^{-1} + p^{-1} < s^{-1} \leq p^{-1}$, and $r^{-1} + s^{-1} < 2n^{-1}$. Then for any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of the Schrödinger operator H there holds

$$|\lambda|^{\alpha - n/2} \leq C\|a\|_r^\alpha \|b\|_s^\alpha, \tag{1.4}$$

where $\alpha := (r^{-1} + s^{-1})^{-1}$, and C being a constant of the potential (it is controlled; see Theorem 4.13). Notice that for the particular case $n = 1, p = 2$ and $r = s = 2$ one has $\alpha = 2$ and $C = 1/2$, and the estimate (1.4) reduces to (1.1). From (1.4) it can be derived estimates for eigenvalues of Schrödinger operators with decaying potentials. So, for instance, taking $a(x) = (1 + |x|^2)^{-\tau/2}$ $(\tau > 0)$, under suitable restrictions on r and τ, for an eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ there holds

$$|\lambda|^{r-n} \leq C \int_{\mathbb{R}^n} |(1 + |x|^2)^{\tau/2} |q(x)|^r \, dx. \tag{1.5}$$

In connection with (1.5) we note the related results obtained in [Fra11] and [Saf10a] (see also [Saf10b] and [DN02]).

Finally, estimates obtained for Schrödinger operators can be successfully extended to polyharmonic operators

$$H_{q,m} = (-\Delta)^m + q,$$

in which (the potential) q is a complex-valued measurable function, and m is an arbitrary positive real number. For the eigenvalues $\lambda \in \mathbb{C} \setminus [0, \infty)$ of an operator of this class it can be proved that

$$|\lambda|^\gamma \leq C \int_{\mathbf{R}^n} |q(x)|^{\gamma + n/2m} \, dx \tag{1.6}$$

for $\gamma > 0$ if $n \geq 2m$ and $\gamma \geq 1 - n/2m$ for $n < 2m$. The estimate given in (1.6) is in fact a result analogous to the mentioned conjecture of Laptev and Safronov [LS09] raised for Schrödinger operators.

The paper consists of five sections: Introductions; Preliminaries. Setting of the problem; Three dimensional Schrödinger operators; Schrödinger operators on $\mathbb{R}^n$; Polyharmonic operators.

2 Preliminaries. Setting of the problem

Consider, in the space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$, the Schrödinger operator

$$-\Delta + q(x) \tag{2.1}$$

with a potential q being in general a complex-valued measurable function on $\mathbb{R}^n$. We assume that the potential q admits a factorization $q = ab$ with a, b belonging to some Lebesgue type spaces (appropriate spaces will be indicated in relevant places). We denote by H_0 the operator defined by $-\Delta$ in $L_p(\mathbb{R}^n)$ on its domain the Sobolev space $W_p^2(\mathbb{R}^n)$, and let A, B denote, respectively, the operators of multiplication by a, b defined in $L_p(\mathbb{R}^n)$ with the maximal domains.

Thus, the differential expression (2.1) defines in the space $L_p(\mathbb{R}^n)$ an operator expressed as the perturbation of H_0 by AB. In order to determine the operator, being a closed extension of $H_0 + AB$, suitable for our purposes, we need to require certain assumptions on the potential. For we let a and b be functions of Stummel classes [Stu56] (see also [JW73] and [Sch86]), namely

$$M_{\nu,p'}(a) < \infty, \quad 0 < \nu < p', \tag{2.2}$$

$$M_{\mu,p}(b) < \infty, \quad 0 < \mu < p, \tag{2.3}$$

(p' is the conjugate exponent to $p : p^{-1} + p'^{-1} = 1$), where it is denoted

$$M_{\nu,p}(u) = \sup_x \int_{|x-y|<1} |u(y)|^p \, |x-y|^{\nu-n} \, dy$$

for functions $u \in L_{p,loc}(\mathbb{R}^n)$. If also the potential q decays at infinity, for instance, like

$$\int_{|x-y|<1} |q(y)| \, dy \to 0 \quad as \quad |x| \to \infty, \tag{2.4}$$

then the operator $H_0 + AB \quad (= -\Delta + q)$ admits a closed extension H having the same essential spectrum as unperturbed operator H_0, i.e.,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) \quad (= \sigma\,(H_0) = [0, \infty)).$$

Note that the conditions (2.2) and (2.3) are used to derived boundedness and also, together with (2.4), compactness domination properties of the perturbation (reasoning are due to Rejto [Rej69] and Schechter [Sch67], cf. also [Sch86]; Theorem 5.1, p.116). To be more precise, due to conditions (2.2) and (2.3), the bordered resolvent $BR(z; H_0)A$ ($R(z; H_0) := (H_0 - zI)^{-1}$ denotes the resolvent of H_0) for some (or, equivalently, any) regular point z of H_0 represents a densely defined operator having a (unique) bounded extension, further on we denote it by $Q(z)$. If, in addition, (2.4), $Q(z)$ is a compact operator and, moreover, it is small with respect to the operator norm for sufficiently large $|z|$.

From now on we let H denote the Schrödinger operator realized in this way in $L_p(\mathbb{R}^n)$ by the differential expression $-\Delta + q(x)$. Notice that constructions related to that mentioned above are widely known in the perturbation theory. In Hilbert case space $p = 2$, H, where the potential q is a real function, represents a self-adjoint operator presenting mainly interest for spectral and scattering problems.

It turns out that there is a constraint relation between the discrete part of the spectrum of H and that of $Q(z)$ (recall $Q(z)$ is the bounded extension of the bordered resolvent $BR(z; H_0)A$), namely, a regular point λ of H_0 is an eigenvalue for the extension H, the Schrödinger operator, if and only if -1 is an eigenvalue of $Q(\lambda)$. This fact, which will play a fundamental role in our arguments, can be deduced essentially, by corresponding accommodation to the situation of Banach space case, using similar arguments as in the proof of Lemma 1 [KK66]. Consequently, for an eigenvalue λ of the Schrödinger operator H, λ being a

regular point of the unperturbed operator H_0, the operator norm of $Q(\lambda)$ must be no less than 1, i.e., $\|Q(\lambda)\| \geq 1$. Namely from this operator norm evaluation we will derive estimates for eigenvalues of the Schrödinger operator H.

Throughout the paper there will always assumed (tacitly) that the conditions (2.2), (2.3) and (2.4) are satisfied.

3 Three dimensional Schrödinger operators

We first consider the case $n = 3$. In this case the fundamental solution of the operator $H_0 - \lambda$ $(= -\Delta - \lambda)$ in $\mathbb{R}^3$, i.e., the solution $\Phi \in S^{'}(\mathbb{R}^3)$ of the equation

$$-(\Delta + \lambda)\Phi(x) = \delta(x), \quad x \in \mathbb{R}^3,$$

is expressed explicitly by

$$\Phi(x) = \frac{1}{4\pi|x|} e^{-\mu|x|}, \quad x \in \mathbb{R}^3,$$

where $\mu = -i\lambda^{1/2}$ and $\lambda^{1/2}$ is chosen so that $\operatorname{Im} \lambda^{1/2} > 0$. Consequently, the resolvent $R(\lambda; H_0) := (H_0 - \lambda)^{-1}$ of H_0 is an integral operator with the kernel

$$\frac{1}{4\pi|x-y|} e^{-\mu|x-y|},$$

that will make useful in evaluation of the bordered resolvent of H_0.

There holds the following result.

Theorem 3.1. *Let $1 < p < \infty$, and let $q = ab$ with $a \in L_r(\mathbb{R}^3)$ and $b \in L_s(\mathbb{R}^3)$ for $0 < r \leq \infty$, $p \leq s \leq \infty$ such that $r^{-1} + s^{-1} < 2/3$. Then, for any eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^3)$, there holds*

$$|\lambda|^{(3-\alpha)/2} \leq C(r, s, \theta) \|a\|_r^\alpha \ \|b\|_s^\alpha, \tag{3.1}$$

where $C(r, s, \theta) = (4\pi)^{1-\alpha}\Gamma(3-\alpha)(\alpha \sin(\theta/2))^{\alpha-3}$ (Γ denotes the gamma function), $\alpha := (1 - r^{-1} - s^{-1})^{-1}$ and $\theta := arg\lambda$ $(\in (0, 2\pi))$.

Proof. We have to show the boundedness of the operator $Q(\lambda) = BR(\lambda; H_0)A$ and evaluate its norm. Note that $Q(\lambda)$ is an integral operator with kernel

$$\frac{1}{4\pi|x-y|} e^{-\mu|x-y|} a(y) b(x).$$

In order to evaluate this integral operator we first observe that, under supposed conditions, the operator of multiplication A is bounded viewed as an operator from $L_p(\mathbb{R}^3)$ to $L_\beta(\mathbb{R}^3)$ with some $\beta \geq 1$. In fact, since $a \in L_r(\mathbb{R}^3)$, for any $u \in L_p(\mathbb{R}^3)$, by Hölder's inequality, we have

$$\|au\|_\beta \leq \|a\|_r \|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}. \tag{3.2}$$

Similarly, one can choose a γ with $p \leq \gamma \leq \infty$, for which

$$\|bv\|_p \leq \|b\|_s \|v\|_\gamma, \quad \gamma^{-1} + s^{-1} = p^{-1}, \tag{3.3}$$

for $v \in L_\gamma(\mathbb{R}^3)$, that means that B represents a bounded operator from $L_\gamma(\mathbb{R}^3)$ to $L_p(\mathbb{R}^3)$.

Now, we observe that the function

$$g(x;\lambda) = \frac{1}{4\pi|x|}e^{-\mu|x|}, \quad x \in \mathbb{R}^3,$$

belongs to the class $L_\alpha(\mathbb{R}^3)$ and, moreover,

$$\|g(\cdot;\lambda)\|_\alpha = (4\pi)^{(1-\alpha)/\alpha}(\alpha\ Re\ \mu)^{(\alpha-3)/\alpha}\ (\Gamma(3-\alpha))^{1/\alpha} \tag{3.4}$$

In fact, by using the polar coordinates $\rho = |x|$, $\omega = x/|x| \in S_2$ (S_2 denotes the unit sphere in $\mathbb{R}^3$), one has

$$\|g(\cdot;\lambda)\|_\alpha^\alpha = \int_{\mathbb{R}^3} \frac{1}{(4\pi|x|)^\alpha} e^{-\alpha(\operatorname{Re}\mu)|x|}\, dx$$

$$= (4\pi)^{-\alpha} \int_0^\infty \int_{S_2} \rho^{-\alpha+2} e^{-\alpha(\operatorname{Re}\mu)\rho}\, d\rho\, d\omega$$

$$= (4\pi)^{-\alpha} mes(S_2) \int_0^\infty \rho^{-\alpha+2} e^{-\alpha(\operatorname{Re}\mu)\rho}\, d\rho.$$

Since $r^{-1} + s^{-1} < 2/3$, that implies $3 - \alpha > 0$, and since $\operatorname{Re}\mu > 0$ (so was chosen μ), the formula (see [GR07]; 3.381.4., p.331)

$$\int_0^\infty x^{\nu-1} e^{-\mu x}\, dx = \mu^{-\nu}\Gamma(\nu), \quad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0 \tag{3.5}$$

can be used, and we find

$$\int_0^\infty \rho^{-\alpha+2} e^{-\alpha(\operatorname{Re}\mu)\rho}\, d\rho = (\alpha \operatorname{Re}\mu)^{\alpha-3}\Gamma(3-\alpha).$$

Since $mes(S_2) = 4\pi$, we finally obtain

$$\|g(\cdot;\lambda)\|_\alpha^\alpha = (4\pi)^{1-\alpha}(\alpha \operatorname{Re}\mu)^{\alpha-3}\Gamma(3-\alpha),$$

i.e., (3.4).

By Young's Inequality (see for instance, [BL76], Theorem 1.2.2, or also [Fol99]; Proposition 8.9a) the operator $R(\lambda; H_0)$, representing an integral operator of convolution type (with the kernel $g(x-y;\lambda)$), is bounded as an operator from $L_\beta(\mathbb{R}^3)$ into $L_\gamma(\mathbb{R}^3)$ provided that

$$\gamma^{-1} + 1 = \alpha^{-1} + \beta^{-1}. \tag{3.6}$$

Moreover,

$$\|R(\lambda; H)v\|_\gamma \leq \|g\|_\alpha \|v\|_\beta, \quad v \in L_\beta(\mathbb{R}^3).$$

Note that (3.6) indeed follows immediately from the relations between p, q, r and s given by (3.2) and (3.3):

$$1-\beta^{-1}+\gamma^{-1}=1-r^{-1}-p^{-1}+p^{-1}-s^{-1}=1-r^{-1}-s^{-1}=\alpha^{-1}.$$

The evaluations (3.2), (3.3) and (3.4) made above imply that

$$\|Q(\lambda)u\|_p=\|BR(\lambda;H_0)Au\|_p\le\|a\|_r\ \|b\|_s\ \|g\|_\alpha\|u\|_p$$

for each $u\in L_p(\mathbb{R}^3)$. Therefore, in view of (3.4), we have

$$\|Q(\lambda)\|\le(4\pi)^{(1-\alpha)/\alpha}(\alpha\operatorname{Re}\mu)^{(\alpha-3)/\alpha}(\Gamma(3-\alpha))^{1/\alpha}\|a\|_r\|b\|_s. \tag{3.7}$$

The desired estimate (3.1) for the eigenvalue λ follows from the fact that the value on the left-hand side of (3.7) must be at least equal to 1 (note that $\operatorname{Re}\mu=\operatorname{Re}(-i\lambda^{1/2})=|\lambda|^{1/2}\sin(\theta/2)$ by letting $\lambda=|\lambda|e^{i\theta},\ 0<\theta<2\pi$). □

From the estimate (3.1) it can be derived many particular estimates useful in applications. We begin with the situation when $a,b\in L_r(\mathbb{R}^3)$ with $r>3$ if $1<p\le 3$ and $p\le r\le\infty$ if $p>3$. In (3.1) we can take $s=r$, then $r^{-1}+s^{-1}=2r^{-1}(<2/3)$ and $\alpha=r/(r-2)$. In view of Theorem 3.1, we have the following result.

Corollary 3.2. *Suppose $q=ab$, where $a,b\in L_r(\mathbb{R}^3)$ with $r>3$ if $1<p\le 3$ and $p\le r\le\infty$ if $p>3$. Then for any eigenvalue $\lambda\in\mathbb{C}\setminus[0,\infty)$ of the Schrödinger operator H, considered in $L_p(\mathbb{R}^3)$, there holds*

$$|\lambda|^{r-3}\le C(r,\theta)\ \|a\|_r^r\ \|b\|_r^r, \tag{3.8}$$

where

$$C(r,\theta)=(4\pi)^{-2}\Gamma(2(r-3)/(r-2))^{r-2}\ ((r-2)/r\sin(\theta/2))^{2(r-3)},$$

in which $\theta=arg\lambda$ $(0<\theta<2\pi)$.

The following estimate was conjectured, but for the case of Hilbert space $L_2(\mathbb{R}^3)$, by Laptev and Safronov [LS09].

Corollary 3.3. *Let $\gamma>0$ for $1<p\le 3$ and $2\gamma\ge p-3$ for $p>3$, supposing that*

$$q\in L_{\gamma+3/2}(\mathbb{R}^3).$$

Then every eigenvalue $\lambda\in\mathbb{C}\setminus[0,\infty)$ of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^3)$, satisfies

$$|\lambda|^\gamma\le C(\gamma,\theta)\int_{\mathbb{R}^3}|q(x)|^{\gamma+3/2}\,dx, \tag{3.9}$$

where

$$C(\gamma,\theta)=(1/4\pi\sin^{2\gamma}(\theta/2))((2\gamma+1)/(2\gamma+3))^{2\gamma}\Gamma(4\gamma/(2\gamma+1))^{(2\gamma+1)/2}.$$

Proof. It suffices to let $r = 2\gamma + 3$ in (3.8) and take

$$a(x) = |q(x)|^{1/2}, \quad b(x) = (\operatorname{sgn} q(x))|q(x)|^{1/2},$$

where sgn $q(x) = q(x)/|q(x)|$ if $q(x) \neq 0$ and sgn $q(x) = 0$ if $q(x) = 0$. □

Frank [Fra11] also obtains a result concerning already mentioned conjecture in case of the Hilbert space $L_2(\mathbb{R}^3)$ and with restriction $0 < r \leq 3/2$. The proofs in [Fra11] are based on a uniform Sobolev inequality due to Kenig, Ruiz and Sogge [KRS87].

Another type of estimates can be obtained directly from (3.8) by involving decaying potentials. So, for instance, if we take $a(x) = (1 + |x|^2)^{-\tau/2}$ with $\tau r > 3$, r satisfying restrictions attributed as in Corollary 3.2, then, by using the formula (see [GR07]; 3.251.2.),

$$\int_0^\infty x^{\mu-1}(1+x^2)^{\nu-1}\, dx = \frac{1}{2}B(\mu/2, 1-\nu-\mu/2), \quad \operatorname{Re}\mu > 0, \quad \operatorname{Re}(\nu+\mu/2) < 1,$$

where $B(x, y)$ denotes the beta-function

$$B(x, y) = \int_0^1 t^x (1-t)^{y-1}\, dt, \quad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,$$

we can calculate

$$\begin{aligned}
\|a\|_r^r &= \int_{\mathbb{R}^3} (1 + |x|^2)^{-\tau r/2}\, dx = \int_0^\infty \int_{S^2} \rho^2 (1+\rho^2)^{-\tau r/2}\, d\rho\, d\omega \\
&= 4\pi \int_0^\infty \rho^2 (1+\rho^2)^{-\tau r/2}\, d\rho = 2\pi B(3/2, \tau r/2 - 3/2),
\end{aligned}$$

and, further, taking $b(x) = (1 + |x|^2)^{\tau/2} q(x)$, we obtain the following result.

Corollary 3.4. *Suppose*

$$(1 + |x|^2)^{\tau/2} q \in L_r(\mathbb{R}^3),$$

where $\tau r > 3$, and $r > 3$ if $1 < p \leq 3$ and $p \leq r \leq \infty$ if $p > 3$. Then every eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$ of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^3)$, satisfies

$$|\lambda|^{r-3} \leq C_1(r, \theta) \int_{\mathbb{R}^3} |(1 + |x|^2)^{\tau/2} q(x)|^r\, dx, \tag{3.10}$$

where $C_1(r, \theta) = 2\pi B(3/2, \tau r/2 - 3/2) C(r, \theta)$, $C(r, \theta)$ being determined as in (3.8).

It stands to reason that estimates of type (3.10) can be given choosing other (weight) functions, used frequently for diverse proposes, as, for instance, $e^{\tau|x|}$, $|x|^\sigma e^{\tau|x|}$, $e^{\tau|x|^2}$, etc.. We cite [Fra11] (see also [Saf10a] and [Saf10b] for some related results involving weight-functions as in Corollary 3.4).

Remark 3.5. The estimate (3.1) can be improved up to a factor $(A_\alpha A_\beta A_{\gamma'})^3$ if in proving of Theorem 3.1 it would be used the sharp form of Young's convolution inequality due to Beckner [Bec75], where A_α, A_β and $A_{\gamma'}$ are defined in accordance with the notation $A_p = (p^{1/p}/p'^{1/p'})^{1/2}$. If it turns out that $A_\alpha A_\beta A_{\gamma'} < 1$ as, for instance, in case $1 < \alpha, \beta, \gamma' < 2$, one has indeed an improvement of (3.1). So it happens in (3.1) for the particular case $r = s = 4$. It can be supposed $q \in L_2(\mathbb{R}^3)$ and, as is easily checked for the possible eigenvalues λ, there holds

$$|\lambda| \leq \frac{1}{64\pi^2 \sin^2(\theta/2)} \|q\|_2^4. \tag{3.11}$$

However, as is seen, $\alpha = 2$, $\beta = \gamma' = 4/3$, hence the constant in (3.11) can be improved by the factor $2^3 \cdot 3^{-3/4}$.

4 Schrödinger operators in $\mathbb{R}^n$

1. If $n > 3$, the method used in the proof of Theorem 3.1 is certainly applicable, actually with major difficulties. For the general case the fundamental solution $\Phi(x)$ of the Laplacian $-\Delta$, and therefore the kernel of the resolvent $R(\lambda; H_0)$, is expressed by Bessel's functions (see, for instance, [BS91]). Of course, the asymptotic formula

$$\Phi(x) = c|x|^{-(n-1)/2} e^{-\mu|x|} \,(1 + o(1)), \quad |x| \to \infty,$$

with $c > 0$ and $\operatorname{Re} \mu > 0$, is useful in that work, however we have not use this fact. Nevertheless, an estimate related to (3.1) can be obtained for the general case $n \geq 3$ by using the following integral representation of the free Green function

$$g(x - y; \lambda) = (4\pi)^{-n/2} \int_0^\infty e^{\lambda t} e^{-|x-y|^2/4t} t^{-n/2} \, dt, \quad \operatorname{Re} \lambda < 0. \tag{4.1}$$

In other words we use the fact that the resolvent $R(\lambda; H_0)$ can be represented as a convolution operator with the kernel (4.1), namely

$$(R(\lambda; H_0)u)(x) = \int_{\mathbb{R}^n} g(x - y; \lambda) u(y) \, dy, \quad \operatorname{Re} \lambda < 0.$$

As in the previous subsection we suppose that the potential q is factorized as $q = ab$, where $a \in L_r(\mathbb{R}^n)$ and $b \in L_s(\mathbb{R}^n)$ with $0 < r \leq \infty$ and $p \leq s < \infty$, and let A, B denote the operators of multiplication by a, b, respectively. By similar arguments to those used in the proof of Theorem 3.1, one can obtain corresponding evaluations for $\mathbb{R}^n$ exactly as (3.2) and (3.3). Accordingly, A can be viewed as a bounded operator from $L_p(\mathbb{R}^n)$ to $L_\beta(\mathbb{R}^n)$ and, respectively, B from $L_\gamma(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, where

$$\beta^{-1} = r^{-1} + p^{-1}, \quad p^{-1} = s^{-1} + \gamma^{-1}. \tag{4.2}$$

Now, we take an $\alpha \geq 1$ such that

$$\alpha^{-1} + \beta^{-1} = \gamma^{-1} + 1 \tag{4.3}$$

and find conditions under which the kernel function $g(x;\lambda)$ belongs to the space $L_\alpha(\mathbb{R}^n)$. By Minkowski's inequality we have

$$\begin{aligned}
\|g(\cdot;\lambda)\|_\alpha &= \left(\int_{\mathbb{R}^n}\left|(4\pi)^{-n/2}\int_0^\infty e^{\lambda t}e^{-|x|^2/4t}\ t^{-n/2}\,dt\right|^\alpha dx\right)^{1/\alpha} \\
&\leq (4\pi)^{-n/2}\int_0^\infty\left(\int_{\mathbb{R}^n}\left|e^{\lambda t}e^{-|x|^2/4t}\ t^{-n/2}\right|^\alpha dx\right)^{1/\alpha} dt \\
&= (4\pi)^{-n/2}\int_0^\infty\left(\int_{\mathbb{R}^n} e^{-\alpha|x|^2/4t}\,dx\right)^{1/\alpha} e^{(\operatorname{Re}\lambda)t}t^{-n/2}\,dt,
\end{aligned}$$

and since

$$\int_{\mathbb{R}^n} e^{-\alpha|x|^2/4t}\,dx = (4\pi t/\alpha)^{n/2},$$

it follows

$$\begin{aligned}
\|g(\cdot;\lambda)\|_\alpha &= (4\pi)^{-n/2}\int_0^\infty (4\pi t/\alpha)^{n/2\alpha}t^{-n/2}e^{(\operatorname{Re}\lambda)t}\,dt \\
&= (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}\int_0^\infty t^{-n/2\alpha'}e^{(\operatorname{Re}\lambda)t}\,dt.
\end{aligned}$$

If α is chosen so that

$$-\frac{n}{2\alpha'} + 1 > 0, \quad i.e. \quad \alpha < \frac{n}{n-2}, \tag{4.4}$$

it can be applied the formula (3.5), and we obtain

$$\|g(\cdot;\lambda)\|_\alpha \leq (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}|\operatorname{Re}\lambda|^{-1+n/2\alpha'}\Gamma(1-n/2\alpha').$$

By Young's inequality we have

$$\|R(\lambda;H_0)v\|_\gamma \leq \|g(\cdot;\lambda)\|_\alpha\ \|v\|_\beta, \quad v\in L_p(\mathbb{R}^n) \tag{4.5}$$

provided that (4.3).

Thus, under supposed conditions, we obtain the following estimation

$$\|Q(\lambda)\| \leq (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}|\operatorname{Re}\lambda|^{-1+n/2\alpha'}\Gamma(1-n/2\alpha')\|a\|_r\ \|b\|_s,$$

and, therefore, for each $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda<0$ such that $\|Q(\lambda)\|\geq 1$, in particular, for an eigenvalue of the Schrödinger operator H, there holds the estimation

$$|\operatorname{Re}\lambda|^{1-n/2\alpha'} \leq (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}\Gamma(1-n/2\alpha')\|a\|_r\ \|b\|_s.$$

Eliminating β and γ with taking into account (4.2) and (4.3), we see that $\alpha = (1-r^{-1}-s^{-1})^{-1}$, and, due to (4.4), with the restriction $r^{-1}+s^{-1}<4n^{-1}$.

We have proved the following result.

Theorem 4.1. *Let $n \geq 3, 1 < p < \infty$ and let $q = ab$, where $a \in L_r(\mathbb{R}^n), b \in L_s(\mathbb{R}^n)$ with $0 < r \leq \infty, p \leq s \leq \infty$ and $r^{-1} + s^{-1} < 2n^{-1}$. Then, for any eigenvalue λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^n)$, there holds*

$$|\operatorname{Re}\lambda|^{1-n/2\alpha'} \leq C(n,r,s)\|a\|_r\|b\|_s, \tag{4.6}$$

where $C(n,r,s) = (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}\Gamma(1-n/2\alpha'),\ \alpha = (1-r^{-1}-s^{-1})^{-1}$.

The following consequences of Theorem 4.1 are natural extensions of the corresponding results given by Corollaries 3.2, 3.3 and 3.4.

Corollary 4.2. *Suppose $q = ab$, where $a \in L_r(\mathbb{R}^n)$ with $r > n$ if $1 < p \leq n$ and $p \leq r \leq \infty$ if $p > n$. Then every eigenvalue λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^n)$, satisfies*

$$|\operatorname{Re}\lambda|^{r-n} \leq C(n,r)\|a\|_r^r\|b\|_r^r, \tag{4.7}$$

where $C(n,r) = (4\pi)^{-n}(1-2r^{-1})^{n(r-2)/2}\Gamma(1-nr^{-1})^r$.

Corollary 4.2 in turn implies the conjuncture for the case $\mathbb{R}^n$ raised by Laptev and Safronov in [LS09]. For it suffices to let $r = 2\gamma + n$ with suitable γ.

Corollary 4.3. *Let $\gamma > 0$ if $1 < p \leq n$ and $2\gamma \geq p - n$ if $p > n$. Suppose*

$$q \in L_{\gamma+n/2}(\mathbb{R}^n).$$

Then every eigenvalue λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^n)$, satisfies

$$|\lambda|^\gamma \leq C(n,\gamma,\theta)\int_{\mathbb{R}^n}|q(x)|^{\gamma+n/2}\,dx, \tag{4.8}$$

where

$$C(n,\gamma,\theta) = \frac{1}{(4\pi)^{n/2}\cos^{2\gamma}\theta}\left(\frac{2\gamma+n-2}{2\gamma+n}\right)^{n(2\gamma+n-2)/4}\Gamma\left(\frac{2\gamma}{2\gamma+n}\right)^{\gamma+n/2},$$

$\pi - \theta = arg\lambda \quad (-\pi/2 < \theta < \pi/2)$.

Next, we let $a(x) = (1+|x|^2)^{-\tau/2}$ and $b(x) = (1+|x|^2)^{\tau/2}q(x)$ for some $\tau > 0$. It is seen that, when $\tau r > n$, one has $a \in L_r(\mathbb{R}^n)$ and, moreover,

$$\|a\|_r^r = \pi^{n/2}\Gamma\left(\frac{\tau r - n}{2}\right)\Big/\Gamma\left(\frac{\tau r}{2}\right).$$

For we apply similar arguments as in the proof of Corollary 3.4 and use the relation between beta and gamma functions (cf. [GR07]; 8.384.1.).

In view of Corollary 4.2 the following result hold true.

Corollary 4.4. *Suppose*

$$(1+|x|^2)^{\tau/2}q \in L_r(\mathbb{R}^n),$$

where $\tau r > 0$, *and* $r > n$ *if* $1 < p \le n$ *and* $p \le r \le \infty$ *if* $p > n$. *Then every eigenvalue* λ *with* $\operatorname{Re}\lambda < 0$ *of the Schrödinger operator* H, *considered acting in* $L_p(\mathbb{R}^n)$, *satisfies*

$$|\operatorname{Re}\lambda|^{r-n} \le C_1(n,r) \int_{\mathbb{R}^n} |(1+|x|^2)^{\tau/2}q(x)|^r \, dx, \tag{4.9}$$

where

$$C_1(n,r) = \pi^{n/2} C(n,r) \Gamma\left(\frac{\tau r - n}{2}\right) \Big/ \Gamma\left(\frac{\tau r}{2}\right),$$

$C(n,r)$ *being determined as in* (4.7).

Remark 4.5. By applying the sharp form of Young's inequality [Bec75] the estimation (4.6) refines by the factor $(A_\alpha A_\beta A_{\gamma'})^n$, where the constant A_α, A_β and $A_{\gamma'}$ are defined like in the Remark 3.5.

The following example is used for the purpose of illustration the above statement, although Schrödinger operators with potentials belonging to the space $L_n(\mathbb{R}^n)$, as is assumed, occur in certain situations important for applications.

Example 4.6. Let $n \ge 3, p = 2$, and suppose $q \in L_n(\mathbb{R}^n)$. Put $r = s = 2n$, and let $a(x) = |q(x)|^{1/2}, b(x) = q(x)/|q(x)|^{1/2}$. Then, by (4.7), for eigenvalues λ with $\operatorname{Re}\lambda < 0$ of H there holds

$$|\operatorname{Re}\lambda| \le C\|q\|_n^2 \tag{4.10}$$

with a constant C depending only on n, namely, $C = 4^{-1}(1-n^{-1})^{n-1}$. However, in this case, $\alpha = n/(n+1)$ and $\beta = \gamma' = 2n/(n+1)$, hence the estimate (4.10) also holds true with the constant $C = n(1-n)^{n-1}/(n+1)^{n+1}$ provided that

$$(A_\alpha A_\beta A_{\gamma'})^n = \frac{2}{\sqrt{n}} \left(\frac{n}{n+1}\right)^{(n+1)/2},$$

as is easily checked. Obviously, $A_\alpha A_\beta A_{\gamma'} < 1$.

2. In the previous argument somewhat it was involved the *heat kernel* associated to the Laplacian on $\mathbb{R}^n$. In fact it could be equivalently used the kernel

$$h(x,y;t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}, \ t > 0, \tag{4.11}$$

representing the (one-parameter) semi-group $e^{-tH_0}(0 \le t < \infty)$. More exactly, e^{-tH_0} is represented by the integral operator with the kernel (4.11), i.e.,

$$(e^{-tH_0}u)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} u(y) \, dy, \quad t > 0. \tag{4.12}$$

The arguments similar to those used in proving Theorem 4.1 can be applied to obtain (under suitable conditions) the estimate

$$\|Be^{-tH_0}Au\|_p \le (4\pi t)^{-n/2\alpha'} \alpha^{-n/2\alpha} \|a\|_r \|b\|_s \|u\|_p, \quad u \in L_p(\mathbb{R}^n).$$

Then, from the formula expressing the resolvent $R(\lambda; H_0)$ as the Laplace transform of the semi-group e^{-tH_0} (see, for instance, [HP74]), i.e.,

$$R(\lambda; H_0) = \int_0^\infty e^{\lambda t} e^{-tH_0}\, dt, \quad \operatorname{Re}\lambda < 0, \tag{4.13}$$

we can further estimate

$$\|BR(\lambda; H_0)A\| \le \int_0^\infty e^{(\operatorname{Re}\lambda)t} \|Be^{-tH_0}A\|\, dt$$
$$\le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} \|a\|_r \ \|a\|_s \int_0^\infty t^{-n/2\alpha'} e^{(\operatorname{Re}\lambda)t}\, dt$$
$$\le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} |\operatorname{Re}\lambda|^{-1+n/2\alpha'}\ \Gamma(1-n/2\alpha')\|a\|_r\|b\|_s,$$

i.e.,

$$\|BR(\lambda; H_0)A\| \le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} |\operatorname{Re}\lambda|^{-1+n/2\alpha'}\ \Gamma(1-n/2\alpha')\|a\|_r\|b\|_s,$$

and, thus, we come to the same estimate as in (4.6).

The next result concerns evaluation of the imaginary part for a complex eigenvalue λ of H.

Theorem 4.7. *Let $n \ge 3$, $1 < p < \infty$, and let $q = ab$, where $a \in L_r(\mathbb{R}^n), b \in L_s(\mathbb{R}^n)$ for r, s satisfying $0 < r \le \infty$, $p \le s \le \infty$, $r^{-1} - s^{-1} = 1 - 2p^{-1}$, $2^{-1} - p^{-1} \le r^{-1} \le 1 - p^{-1}$ and $r^{-1} + s^{-1} < 2n^{-1}$. Then, for any complex eigenvalue λ with $\operatorname{Im}\lambda \ne 0$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^n)$, there holds*

$$|\operatorname{Im}\lambda|^\alpha \le (4\pi)^{\alpha-1}\Gamma(\alpha)\|a\|_r\|b\|_s, \tag{4.14}$$

where $\alpha = 1 - n(r^{-1} + s^{-1})/2$.

Proof. The proof will depend upon a modification of the argument used in proving the previous result. Instead of (4.13) it will be used the formula expressing the resolvent $R(\lambda; H_0)$ as the Laplace transform of the operator-group e^{-itH_0} $(-\infty < t < \infty)$, namely

$$R(\lambda; H_0) = i\int_0^\infty e^{i\lambda t} e^{-itH_0}\, dt \tag{4.15}$$

if, for instance, $\operatorname{Im}\lambda > 0$. First, we estimate the norm $\|Be^{-itH_0}A\|$ and then by using the formula (4.15) we will derive estimation for $\operatorname{Im}\lambda$ (we preserve notations made above).

As is known (cf., for instance, [Pro64] and also [Kat95], Ch.IX), for a fixed real t, e^{-itH_0} represents an integral operator with the heat kernel (cf. (4.11))

$$h(x, y; it) = (4\pi it)^{-n/2} e^{-|x-y|^2/4it}.$$

Writing

$$(e^{-itH_0}Au)(x) = (4\pi it)^{-n/2}e^{-|x|^2/4it}\int_{\mathbb{R}^n} e^{-i\langle x,y\rangle/2t}e^{-|y|^2/4it}a(y)u(y)\,dy, \tag{4.16}$$

we argue as follows.

We already know that

$$\|Au\|_\beta \le \|a\|_r\|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}.$$

It follows that for any $u \in L_p(\mathbb{R}^n)$ the function v defined by $v(y) = e^{-|y|^2/4it}a(y)u(y)$ belongs to $L_\beta(\mathbb{R}^n)$, and

$$\|v\|_\beta \le \|a\|_r\|u\|_p. \tag{4.17}$$

Further, the integral on the right-hand side in (4.16) represents the function $(2\pi)^{n/2}\hat{v}(x/2t)$, where $\hat{v}$ denotes the Fourier transform of v. According to the Hausdorf-Young theorem (see, for instance, [BL76], Theorem 1.2.1) the Fourier transform represents a bounded operator from $L_\beta(\mathbb{R}^n)$ to $L_{\beta'}(\mathbb{R}^n)$ with $1 \le \beta \le 2$, and its norm is bounded by $(2\pi)^{-n/2+n/\beta^{'}}$, i.e.,

$$\|\hat{v}\|_{\beta'} \le (2\pi)^{-n/2+n/\beta^{'}}\|v\|_\beta. \tag{4.18}$$

It follows that $\hat{v} \in L_{\beta'}(\mathbb{R}^n)$ and, since

$$(e^{-itH_0}Au)(x) = (4\pi it)^{-n/2}e^{-|x|^2/4it}(2\pi)^{n/2}\hat{v}(x/2t),$$

the function $e^{-itH_0}Au$ belongs to $L_{\beta'}(\mathbb{R}^n)$. Moreover, in view of (4.17) and (4.18),

$$\|e^{-itH_0}Au\|_{\beta'} = (4\pi t)^{-n/2}(2\pi)^{n/2}\left(\int_{\mathbb{R}^n} |\hat{v}(x/2t)|^{\beta^{'}}\,dx\right)^{1/\beta^{'}}$$

$$= (4\pi t)^{-n/2}(2\pi)^{n/2}(2t)^{n/\beta^{'}}\|\hat{v}\|_{\beta'} \le (4\pi t)^{-n/2}(2\pi)^{n/2}(2t)^{n/\beta^{'}}(2\pi)^{-n/2+n/\beta^{'}}\|v\|_\beta$$

$$\le (4\pi t)^{-n/2+n/\beta^{'}}\|a\|_r\ \|u\|_p,$$

so that

$$\|e^{-itH_0}Au\|_{\beta'} \le (4\pi t)^{-n/2+n/\beta^{'}}\|a\|_r\|u\|_p,\ u \in L_p(\mathbb{R}^n).$$

On the other hand, since $r^{-1} - s^{-1} = 1 - 2p^{-1}$, and since $\beta^{-1} = r^{-1} + p^{-1}$, one has $s^{-1} + \beta^{'-1} = p^{-1}$, that guarantees the boundedness of the operator of multiplication B regarded as an operator acting from $L_{\beta'}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$. Moreover,

$$\|Bv\|_p \le \|b\|_s\|v\|_{\beta'}, \quad v \in L_{p'}(\mathbb{R}^n),$$

It is seen that for any $u \in L_p(\mathbb{R}^n)$ the element $Be^{-itH_0}Au$ belongs to $L_p(\mathbb{R}^n)$, and

$$\|Be^{-itH_0}Au\|_p \le (4\pi t)^{-n/2+n/\beta^{'}}\|a\|_r\|b\|_s\|u\|_p, \quad u \in L_p(\mathbb{R}^n).$$

Now, we apply (4.15) and for $\operatorname{Im}\lambda > 0$ we find

$$\|BR(\lambda; H_0)Au\|_p \leq \int_0^\infty e^{-(\operatorname{Im}\lambda)t}\|Be^{-itH_0}Au\|_p \, dt$$

$$\leq (4\pi)^{-n/2+n/\beta'}\|a\|_r\|b\|_s\|u\|_p \int_0^\infty t^{-n/2+n/\beta'} e^{-(\operatorname{Im}\lambda)t} \, dt.$$

Next, we observe $1 - n/2 + n/\beta' = \alpha$ that was assumed to be positive, and thus we can apply the formula (3.5), due to of which, we have

$$\int_0^\infty t^{-n/2+n/\beta'} e^{-(\operatorname{Im}\lambda)t} \, dt = (\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha).$$

Therefore,

$$\|BR(\lambda; H_0)A\| \leq (4\pi)^{\alpha-1}(\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha)\|a\|_r\|b\|_s.$$

For an eigenvalue λ of H it should be

$$1 \leq (4\pi)^{\alpha-1}(\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha)\|a\|_r \; \|b\|_s,$$

that is (4.14).

The estimate for the case $\operatorname{Im}\lambda < 0$ is treated similarly coming from the formula

$$R(\lambda; H_0) = -i\int_\infty^0 e^{i\lambda t}e^{-iH_0 t} \, dt, \quad \operatorname{Im}\lambda < 0.$$

□

Notice that if $r = s$ in Theorem 4.7, it must be only $p = 2$ and $r > n$. For this case we have the following result.

Corollary 4.8. *Let $n \geq 3$, $r > n$, and suppose $q \in L_{r/2}(\mathbb{R}^n)$. Then any complex eigenvalue λ with $Im\lambda \neq 0$ of the Schrödinger operator H defined in the space $L_2(\mathbb{R}^n)$ satisfies*

$$|\operatorname{Im}\lambda|^{1-n/r} \leq (4\pi)^{-n/r}\Gamma(1 - n/r)\|q\|_{r/2}. \tag{4.19}$$

For the particular case when $r = 2\gamma + n$ we have the following result (an analogous result to that given by Corollary 4.2).

Corollary 4.9. *Let $n \geq 3, \gamma > 0$ and suppose that $q \in L_{\gamma+n/2}(\mathbb{R}^n)$. Then for any complex eigenvalue λ with $\operatorname{Im}\lambda \neq 0$ of the Schrödinger operator defined in $L_2(\mathbb{R}^n)$ there holds*

$$|\operatorname{Im}\lambda|^\gamma \leq (4\pi)^{-n/2}\Gamma\left(\frac{2\gamma}{2\gamma+n}\right)^{\gamma+n/2}\int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2} \, dx. \tag{4.20}$$

Remark 4.10. The estimate given by Theorem 4.7 can be improved upon a constant less than 1. The point is that in proving Theorem 4.7 it can be applied the sharp form of the Hausdorff-Young theorem which is due to K. I. Babenko [Bab61] (see also W. Beckner [Bec75] for the general case relevant for our purposes). According to Babenko's result estimation (4.18), and hence (4.14) as well, can be refined upon a constant less than 1, namely

$$\|\hat{v}\|_{\beta'} \leq (2\pi)^{-n/2+n/\beta'} A\|v\|_{\beta},$$

where $A = (\beta^{1/\beta}/\beta'^{1/\beta'})^{n/2}$. It is always $A \leq 1$ provided of $1 \leq \beta \leq 2$, and it is strictly less than 1 if β is chosen such that $1 < \beta < 2$. The same concerns estimates (4.19) and (4.20).

3. The norm evaluation for the operators $BR(\lambda; H_0)A$ for $\lambda \in \mathbb{C}\setminus[0,\infty)$ can be carried out representing the resolvent of H_0 in terms of the Fourier transform. Namely, it can use the following equality

$$BR(\lambda; H_0)A = BF^{-1}\widehat{R(\lambda; H_0)}FA, \tag{4.21}$$

where it is denoted

$$\widehat{R(\lambda; H_0)} = FR(\lambda; H_0)F^{-1}$$

(F, F^{-1} denote the Fourier operators). Clearly, $\widehat{R(\lambda; H_0)}$ represents the multiplication operator by $(|\xi|^2 - \lambda)^{-1}$, i.e.,

$$\widehat{R(\lambda; H_0)}\hat{u}(\xi) = (|\xi|^2 - \lambda)^{-1}\hat{u}(\xi), \quad \xi \in \mathbb{R}^n.$$

On $L_2(\mathbb{R}^n)$ the mentioned relations are obviously true. However, we will use them for the spaces $L_p(\mathbb{R}^n)$ with $p \neq 2$, as well, preserving the same notations as in the Hilbert space case $p = 2$.

As before, by assuming that $a \in L_r(\mathbb{R}^n)$ and $b \in L_s(\mathbb{R}^n)$ $(0 < r, s \leq \infty)$, we choose $\beta > 0$ and $\gamma > 0$ such that

$$\|Au\|_{\beta} \leq \|a\|_r \, \|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}, \tag{4.22}$$

$$\|Bv\|_p \leq \|b\|_s \, \|v\|_{\gamma}, \qquad p^{-1} = s^{-1} + \gamma^{-1}. \tag{4.23}$$

According to the Hausdorff-Young theorem, if $1 \leq \beta \leq 2$, the Fourier transform F represents a bounded operator from $L_{\beta}(\mathbb{R}^n)$ to $L_{\beta'}(\mathbb{R}^n)$ the norm of which is bounded by $(2\pi)^{-n/2+n/\beta'}$, i.e.,

$$\|Ff\|_{\beta'} \leq (2\pi)^{-n/2+n/\beta'}\|f\|_{\beta}. \tag{4.24}$$

The same concerns the inverse Fourier transform F^{-1} considered as an operator acting from $L_{\gamma'}(\mathbb{R}^n)$ to $L_{\gamma}(\mathbb{R}^n)$. If $1 \leq \gamma' \leq 2$, that is equivalent to $2 \leq \gamma \leq \infty$, we have

$$\|F^{-1}g\|_{\gamma} \leq (2\pi)^{-n/2+n/\gamma}\|g\|_{\gamma'}. \tag{4.25}$$

Now, we take $\alpha, 0 < \alpha \leq \infty$, such that

$$\gamma'^{-1} = \alpha^{-1} + \beta'^{-1}, \tag{4.26}$$

equivalently, $\alpha^{-1} = r^{-1} + s^{-1}$, and evaluate the L_α-norm of the function $h(\cdot;\lambda)$ defined by

$$h(\xi;\lambda) = (|\xi|^2 - \lambda)^{-1}, \quad \xi \in \mathbb{R}^n.$$

For $\alpha \neq \infty$ we have

$$\|h(\cdot;\lambda)\|_\alpha^\alpha = \int_{\mathbb{R}^n} ||\xi|^2 - \lambda|^{-\alpha}\, d\xi = \int_0^\infty \int_{S^{n-1}} \rho^{n-1}|\rho^2 - \lambda|^{-\alpha}\, d\rho\, d\omega?$$
$$= mes(S^{n-1}) \int_0^\infty \rho^{n-1}|\rho^2 - \lambda|^{-\alpha}\, d\rho,$$

where $mes(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$ is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. Therefore,

$$\|h(\cdot;\lambda)\|_\alpha^\alpha = 2\pi^{n/2}/\Gamma(n/2) \int_0^\infty \rho^{n-1}|\rho^2 - \lambda|^{-\alpha}\, d\rho. \tag{4.27}$$

If, we are particularly interesting in estimation of negative eigenvalues, we let that $\operatorname{Re}\lambda < 0$ and evaluate the integral in (4.27) as follows. First we observe that

$$|\rho^2 - \lambda|^{-1} \leq (\rho^2 - \operatorname{Re}\lambda)^{-1},$$

and then by setting $\rho^2 = t$ we obtain

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \leq \frac{\pi^{n/2}|\operatorname{Re}\lambda|^{-\alpha}}{\Gamma(n/2)} \int_0^\infty \frac{t^{n/2-1}}{(|\operatorname{Re}\lambda|^{-1}t + 1)^\alpha}\, dt.$$

By supposing $\alpha > n/2$ the formula ([GR07], 3.194.3.)

$$\int_0^\infty \frac{x^{\mu-1}}{(1+\beta x)^\nu}\, dx = \beta^{-\mu} B(\mu, \nu - \mu), \qquad |arg\beta| < \pi, \quad \operatorname{Re}\nu > \operatorname{Re}\mu > 0$$

($B(x,y)$ denotes the beta function), can be applied. We get

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \leq \pi^{n/2}(\Gamma(n/2))^{-1}|\operatorname{Re}\lambda|^{n/2-\alpha} B(n/2, \alpha - n/2),$$

or, in view of the functional relation between beta and gamma functions,

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \leq \pi^{n/2}|\operatorname{Re}\lambda|^{n/2-\alpha}\ \Gamma(\alpha - n/2)/\Gamma(\alpha). \tag{4.28}$$

Thus, for $\alpha > n/2$ the function $h(\cdot;\lambda)$ belongs to the space $L_\alpha(\mathbb{R}^n)$ and, since (4.26), it follows that the operator of multiplication $\widehat{R(\lambda; H_0)}$ is bounded as an operator acting from $L_{\beta'}(\mathbb{R}^n)$ to $L_{\gamma'}(\mathbb{R}^n)$, and, due to of (4.28), there holds

$$\|\widehat{R(\lambda; H_0)} f\|_{\gamma'} \leq \pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha - 1}\ (\Gamma(\alpha - n/2)/\Gamma(\alpha))^{1/\alpha}\|f\|_{\beta'}. \tag{4.29}$$

In this way we obtain (cf. (4.22) - (4.25), (4.29))

$$\|BR(\lambda;H_0)A\| \le (2\pi)^{-n/\alpha}\pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha-1}\ (\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s.$$

Therefore, for an eigenvalue λ of H, it should by fulfilled

$$1 \le (2\pi)^{-n/\alpha}\pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha-1}/(\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s,$$

or, equivalently,

$$|\operatorname{Re}\lambda|^{1-n/2\alpha} \le (4\pi)^{-n/2\alpha}(\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s. \tag{4.30}$$

In the extremal case $\alpha=\infty$, that is only happen if $r=s=\infty$ (recall that $\alpha^{-1}=r^{-1}+s^{-1}$), there holds

$$\|h(\cdot;\lambda)\|_\infty = \sup_{\xi\in\mathbb{R}^n}||\xi|^2-\lambda|^{-1} \le \sup_{\rho>0}(\rho^2-\operatorname{Re}\lambda)^{-1} = |\operatorname{Re}\lambda|^{-1},$$

i.e.,

$$\|h(\cdot;\lambda)\|_\infty \le |\operatorname{Re}\lambda|^{-1}.$$

In accordance with this evaluation, one follows

$$|\operatorname{Re}\lambda| \le \|a\|_\infty\|b\|_\infty, \tag{4.31}$$

a natural estimate for eigenvalues occurred outside of the continuous spectrum of H_0 by bounded perturbations.

Note that the restriction $1\le\beta\le 2$ is equivalent to $2^{-1}-p^{-1}\le r^{-1}\le 1-p^{-1}$, while $1\le\gamma'\le 2$ to $2^{-1}+p^{-1}\le s^{-1}\le p^{-1}$, and $\alpha>n/2$ to $r^{-1}+s^{-1}<2n^{-1}$.

We have proved the following result.

Theorem 4.11. *Let $1<p<\infty$, and let $q=ab$, where $a\in L_r(\mathbb{R}^n)$, $b\in L_s(\mathbb{R}^n)$ for r,s satisfying $0<r\le\infty$, $0<s\le\infty$, $2^{-1}-p^{-1}\le r^{-1}\le 1-p^{-1}$, $-2^{-1}+p^{-1}\le s^{-1}\le p^{-1}$, and $r^{-1}+s^{-1}<2n^{-1}$. Then, for any eigenvalue λ with $\operatorname{Re}\lambda<0$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^n)$, there holds*

$$|\operatorname{Re}\lambda|^{\alpha-n/2} \le C(n,\alpha)\|a\|_r^\alpha\ \|b\|_s^\alpha, \tag{4.32}$$

where $C(n,\alpha)=(4\pi)^{-n/2}\Gamma(\alpha-n/2)/\Gamma(\alpha)$, $\alpha=(r^{-1}+s^{-1})^{-1}$.

For $r=s=\infty$ there holds (4.31).

For the particular case $n=1,p=2$ and $r=s=2$ one has $\alpha=1$ and $C=1/2$, hence, in view of (4.32), the following estimate

$$|\operatorname{Re}\lambda|^{1/2} \le \frac{1}{2}\|V\|_1\ \left(=\frac{1}{2}\int_{-\infty}^{\infty}|V(x)|\,dx\right) \tag{4.33}$$

holds true for any eigenvalue λ of H with $\operatorname{Re}\lambda<0$.

The obtained evaluation (4.33) corresponds to the well-known result of L. Spruch (mentioned in [Kel61]) concerning negative eigenvalues of the one-dimensional self-adjoint Schrödinger operator considered in $L_2(\mathbb{R})$. For other related results see [AAD01], [DN02], [FLS11], [FLLS06], [LS09] and [Saf10a].

Theorem 4.11 implies more general result (cf. also Corollary 4.3).

Corollary 4.12. *Let $\gamma > 0$ for $n \geq 2$ and $\gamma \geq 1/2$ for $n = 1$. If $q \in L_{\gamma+n/2}(\mathbb{R}^n)$, then every eigenvalue λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H defined in $L_2(\mathbb{R}^n)$ satisfies*

$$|\operatorname{Re}\lambda|^\gamma \leq (4\pi)^{-n/2}\frac{\Gamma(\gamma)}{\Gamma(\gamma+n/2)}\int_{\mathbb{R}^n}|q(x)|^{\gamma+n/2}\,dx. \tag{4.34}$$

A rigorous evaluation of the integral on the right-hand side of (4.27) leads to more exact estimates for the perturbed eigenvalues. To this end, we let $\lambda = |\lambda|e^{i\theta}(0 < \theta < 2\pi)$ and put $\rho^2 = |\lambda|t$. Then

$$\int_0^\infty \frac{\rho^{n-1}}{|\rho^2-\lambda|^\alpha}\,d\rho = \frac{1}{2}|\lambda|^{n/2-\alpha}\int_0^\infty \frac{t^{n/2-1}}{(t^2-2t\cos\theta+1)^{\alpha/2}}\,dt.$$

If $n/2 < \alpha$, it can be applied the formula ([GR07]; 3.252.10.)

$$\int_0^\infty \frac{x^{\mu-1}}{(x^2+2x\cos t+1)^\nu}\,dx = (2\sin t)^{\nu-1/2}\Gamma(\nu+1/2)B(\mu,2\nu-\mu)P^{1/2-\nu}_{\mu-\nu-1/2}(\cos t)$$

$$(-\pi < t < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}\ 2\nu),$$

where $P^\nu_\mu(z)(-1 \leq z \leq 1)$ denote for the spherical harmonics of the first kind ([GR07]; 8.7 - 8.8). As a result we have

$$\|h(\cdot;\lambda)\|_\alpha = \pi^{n/2\alpha}|\lambda|^{n/2\alpha-1}I(n,\alpha,\theta), \tag{4.35}$$

where

$$I(n,\alpha,\theta) = (2\sin\theta)^{1/2-1/2\alpha}\left(\frac{\Gamma(\alpha/2+1/2)\Gamma(\alpha-n/2)}{\Gamma(\alpha)}P^{1/2-\alpha/2}_{n/2-\alpha/2-1/2}(-\cos\theta)\right)^{1/2},$$

and hence

$$\|BR(\lambda;H_0)Au\|_p \leq (4\pi^{-n/2\alpha})|\lambda|^{n/2\alpha-1}I(n,\alpha,\theta)\|a\|_r\|b\|_s\|u\|_p$$

(note that $(-n/2+n/\beta') + (-n/2+n/\gamma) + n/2\alpha = -n/2\alpha$).

Therefore, we obtain the following result.

Theorem 4.13. *Under the same assumptions as in Theorem 4.11 for any eigenvalue $\lambda \in \mathbb{C}\setminus[0,\infty)$ of the Schrödinger operator H, considering acting in the space $L_p(\mathbb{R}^n)$, there holds the estimation*

$$|\lambda|^{\alpha-n/2} \leq C(n,\alpha,\theta)\|a\|^\alpha_r\|b\|^\alpha_s, \tag{4.36}$$

where $C(n,\alpha,\theta) = (4\pi)^{-n/2}I(n,\alpha,\theta)^\alpha$ and $I(n,\alpha,\theta)$ as in (4.35).

Remark 4.14. The estimate (4.32) and, of course, (4.36) as well can be improved upon the constant $A_\beta A_{\gamma'}(=(\beta^{1/\beta}\gamma'^{1/\gamma'}/\beta'^{1/\beta'}\gamma^{1/\gamma})^{n/2})$ due to the sharp form of the Hausdorff-Young theorem [Bab61] (cf. Remark 4.10).

5 Polyharmonic operators

We will extend the estimates established previously to the operators of the form

$$H = (-\Delta)^m + q$$

in which (the potential) q is a complex-valued function, and m is an arbitrary positive real number. Unperturbed operator

$$H_0 = (-\Delta)^m$$

can be comprehend, as

$$(H_0 u)(x) = \int_{\mathbb{R}^n} |\xi|^{2m} \hat{u}(\xi) e^{-i\langle x,\xi\rangle}\, d\xi$$

defined, for instance, in $L_2(\mathbb{R}^n)$ on its maximal domain consisting of all functions $u \in L_2(\mathbb{R}^n)$ such that $H_0 u \in L_2(\mathbb{R}^n)$ (or, what is the same, $\hat{v}$ determined by $\hat{v}(\xi) = |\xi|^{2m}\hat{u}(\xi)$ belongs to $L_2(\mathbb{R}^n)$); $\hat{u}$ denotes the Fourier transform of u. H_0 can be treated upon a unitary equivalence (by the Fourier transform) as the operator of multiplication by $|\xi|^{2m}$.

In the space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$ the operator H can be viewed as an elliptic operator of order $2m$ defined on its domain the Sobolev space $W_p^{2m}(\mathbb{R}^n)$. As in preceding sections we assume that the potential q admits a factorization $q = ab$ with a, b for which conditions (2.2), (2.3), but with $0 < \nu < p^{'}\kappa$ and $0 < \mu < p(m - \kappa)$ for some $0 < \kappa < m$, and (2.4) are satisfied. Under these conditions the operator $(-\Delta)^m + q$ admits a closed extension H, let us denote it by $H_{m,q}$, to which the approach for the evaluation of perturbed eigenvalues proposed in Section 2 is applied.

Thus, in order to obtain estimation for the norm of $BR(\lambda; H_0)A$ (the operators A, B are defined as in previous subsections), we can use the relation (4.21), where

$$\widehat{R(\lambda; H_0)}\hat{u}(\xi) = (|\xi|^{2m} - \lambda)^{-1}\hat{u}(\xi), \quad \xi \in \mathbb{R}^n.$$

The arguments used in proving Theorems 4.11 and 4.13 can be applied, and as is seen we have only to evaluate, for appropriate $\alpha > 0$, the L_α-norm of the function $h_m(\cdot;\lambda)$ defined by

$$h_m(\xi;\lambda) = (|\xi|^{2m} - \lambda)^{-1}, \quad \xi \in \mathbb{R}^n.$$

For any $\alpha, 0 < \alpha < \infty$, we have

$$\|h_m(\cdot;\lambda)\|_\alpha^\alpha = \int_{\mathbb{R}^n} \frac{d\xi}{||\xi|^{2m} - \lambda|^\alpha} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{\rho^{n-1}}{|\rho^{2m} - \lambda|^\alpha}\, d\rho.$$

Writing $\lambda = |\lambda| e^{i\theta} (0 < \theta < 2\pi)$ and making the substitution $\rho^{2m} = |\lambda| t$, we obtain

$$\int_0^\infty \frac{\rho^{n-1}}{|\rho^{2m} - \lambda|^\alpha}\, d\rho = \frac{1}{2m}|\lambda|^{n/2m-\alpha} \int_0^\infty \frac{t^{n/2m-1}}{(t^2 - 2t\cos\theta + 1)^{\alpha/2}}\, dt.$$

Assuming $n/2m < \alpha$ we apply again the formula ([GR07]; 3.252.10.), and obtain Hence,

$$\|h_m(\cdot;\lambda)\|_\alpha^\alpha = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{1}{2m} |\lambda|^{n/2m-\alpha} I_m(n,\alpha,\theta), \tag{5.1}$$

where

$$I_m(n,\alpha,\theta) = (2\sin\theta)^{\alpha/2-1/2}\Gamma(\alpha/2+1/2)B(n/2m, \alpha-n/2m)P^{1/2-\alpha/2}_{n/2m-\alpha/2-1/2}(-\cos\theta).$$

Collecting all evaluations we obtain the following result.

Theorem 5.1. *Let* $1 < p < \infty, m > 0$, *and let* $q = ab$, *where* $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ *for* r, s *satisfying* $0 < r \leq \infty$, $0 < s \leq \infty$, $2^{-1} - p^{-1} \leq r^{-1} \leq 1 - p^{-1}$, $-2^{-1} + p^{-1} \leq s^{-1} \leq p^{-1}$, *and* $r^{-1} + s^{-1} < 2mn$. *Then, for any eigenvalue* $\lambda \in \mathbb{C} \setminus [0,\infty)$ *of the operator* $H_{m,q}$, *considered acting in* $L_p(\mathbb{R}^n)$, *there holds*

$$|\lambda|^{\alpha-n/2m} \leq C(n,m,\alpha,\theta)\|a\|_r^\alpha \|b\|_s^\alpha, \tag{5.2}$$

where $C(n,m,\alpha,\theta) = (4\pi)^{-n/2}(m\Gamma(n/2))^{-1} I_m(n,\alpha,\theta)$, $I_m(n,\alpha,\theta)$ *is determined as in* (5.1), *and* $\alpha = (r^{-1}+s^{-1})^{-1}$.

As a consequence of Theorem 5.1 we have a result analogous to that given by Corollary 4.12.

Corollary 5.2. *Let* $\gamma > 0$ *for* $n \geq 2m$ *and* $\gamma \geq 1 - n/2m$ *for* $n < 2m$. *If* $q \in L_{\gamma+n/2m}(\mathbb{R}^n)$, *then every eigenvalue* $\lambda \in \mathbb{C} \setminus [0,\infty)$ *of the operator* $H_{m,q}$ *defined in* $L_2(\mathbb{R}^n)$ *satisfies*

$$|\lambda|^\gamma \leq C(n,m,\alpha,\theta) \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2m}\, dx, \tag{5.3}$$

where $C(n,m,\alpha,\theta)$ *is as in (2.54).*

Remark 5.3. Similarly, as for estimates (4.32) and (4.36), the estimate (5.2) and hence (5.3) can be improved upon the constant $A_\beta A_{\gamma'}$ (see Remark 4.14).

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References

[AAD01] A. A. Abramov, A. Aslanyan, and E. B. Davies. Bounds on complex eigenvalues and resonances. *J. Phys. A*, 34(1):57–72, 2001.

[Bab61] K. I. Babenko. An inequality in the theory of Fourier integrals. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:531–542, 1961.

[Bec75] W. Beckner. Inequalities in Fourier analysis. *Ann. of Math. (2)*, 102(1):159–182, 1975.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[BS91] F. A. Berezin and M. A. Shubin. *The Schrödinger equation*, volume 66 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1991.

[Dav02] E. B. Davies. Non-self-adjoint differential operators. *Bull. London Math. Soc.*, 34(5):513–532, 2002.

[DN02] E. B. Davies and Jiban Nath. Schrödinger operators with slowly decaying potentials. *J. Comput. Appl. Math.*, 148(1):1–28, 2002.

[FLLS06] R. L. Frank, A. Laptev, E. H. Lieb, and R. Seiringer. Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials. *Lett. Math. Phys.*, 77(3):309–316, 2006.

[FLS11] R. L. Frank, A. Laptev, and R. Seiringer. A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials. In *Spectral theory and analysis*, volume 214 of *Oper. Theory Adv. Appl.*, pages 39–44. Birkhäuser/Springer Basel AG, Basel, 2011.

[Fol99] G. B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999.

[Fra11] R. L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.*, 43(4):745–750, 2011.

[GR07] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[HP74] E. Hille and R. S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society, Providence, R. I., 1974.

[JW73] K. Jörgens and J. Weidmann. *Spectral properties of Hamiltonian operators.* Lecture Notes in Mathematics, Vol. 313. Springer-Verlag, Berlin, 1973.

[Kat95] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kel61] J. B. Keller. Lower bounds and isoperimetric inequalities for eigenvalues of the Schrödinger equation. *J. Mathematical Phys.*, 2:262–266, 1961.

[KK66] R. Konno and S. T. Kuroda. On the finiteness of perturbed eigenvalues. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:55–63 (1966), 1966.

[KRS87] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.

[LS09] A. Laptev and O. Safronov. Eigenvalue estimates for Schrödinger operators with complex potentials. *Comm. Math. Phys.*, 292(1):29–54, 2009.

[Pro64] R. T. Prosser. Convergent perturbation expansions for certain wave operators. *J. Mathematical Phys.*, 5:708–713, 1964.

[Rej69] P. A. Rejto. On partly gentle perturbations. III. *J. Math. Anal. Appl.*, 27:21–67, 1969.

[Saf10a] O. Safronov. Estimates for eigenvalues of the Schrödinger operator with a complex potential. *Bull. Lond. Math. Soc.*, 42(3):452–456, 2010.

[Saf10b] O. Safronov. On a sum rule for Schrödinger operators with complex potentials. *Proc. Amer. Math. Soc.*, 138(6):2107–2112, 2010.

[Sch67] M. Schechter. Essential spectra of elliptic partial differential equations. *Bull. Amer. Math. Soc.*, 73:567–572, 1967.

[Sch86] M. Schechter. *Spectra of partial differential operators*, volume 14 of *North-Holland Series in Applied Mathematics and Mechanics*. North-Holland Publishing Co., Amsterdam, second edition, 1986.

[Stu56] F. Stummel. Singuläre elliptische Differential-operatoren in Hilbertschen Räumen. *Math. Ann.*, 132:150–176, 1956.